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ANALYSIS OF TWO-DIMENSIONAL DIFFUSION-CONTROLLED MOVING BOUNDARY PROBLEMS

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Abstract—This paper presents a technique for the analysis of unsteady, two-dimensional diffusive heat- or mass-transfer problems characterized by moving irregular boundaries. The technique includes an immobilization transformation and a numerical scheme for the solution of the transformed equations. Specifically, the immobilization consists of transforming the governing partial differential equations into a coordinate system where the phase boundaries correspond to fixed coordinate surfaces. An example problem involving the solidification or melting of a finite cylinder is analyzed, and results for a range of conditions are presented.

NOMENCLATURE

$\hat{C}_p$, specific heat capacity at constant pressure;
H^*, position of phase boundary;
H, dimensionless phase boundary position, $= \dfrac{H^*}{R}$;
H_0^*, initial position of phase boundary;
H_0, dimensionless initial phase boundary position, $= \dfrac{H_0^*}{R}$;
$\Delta\hat{H}$, specific enthalpy of lower phase minus specific enthalpy of upper phase;
h, heat-transfer coefficient to surroundings;
$I_n(x)$, modified Bessel function of first kind of order n;
k, thermal conductivity;
L, length of cylinder;
M, number of radial difference increments in each phase;
N, number of axial difference increments in each phase;
R, radius of cylinder;
r^*, radial coordinate;
r, dimensionless radial coordinate, $= \dfrac{r^*}{R}$;
T^*, temperature;
T, dimensionless temperature, $= \dfrac{T^* - T_1}{T_2 - T_1}$;
T_1, temperature at top of cylinder;
T_2, temperature at bottom of cylinder;
T_3, temperature of surroundings;
T_f, fusion temperature;
t^*, time;
t, dimensionless time, $= \dfrac{\alpha t^*}{R^2}$;
z^*, axial variable;
z, dimensionless axial variable, $= \dfrac{z^*}{R}$.

Greek symbols

α, thermal diffusivity, $= \dfrac{k}{\rho \hat{C}_p}$;
β, aspect ratio, $= \dfrac{L}{R}$;
γ, Nusselt number, $= \dfrac{hR}{k}$;
δ, dimensionless fusion temperature, $= \dfrac{T_f - T_1}{T_2 - T_1}$;
ε, dimensionless temperature of surroundings, $= \dfrac{T_3 - T_1}{T_2 - T_1}$;
ζ, ratio of thermal diffusivities, $= \dfrac{\bar{\alpha}}{\alpha}$;
θ, ratio of latent heat to sensible heat, $= \dfrac{\Delta\hat{H}}{\hat{C}_p(T_2 - T_1)}$;
κ, thermal conductivity ratio, $= \dfrac{\bar{k}}{k}$;
λ, dimensionless independent variable, $= r$;
ξ, dimensionless independent variable defined by equation (21);

$\bar{\xi}$, dimensionless independent variable defined by equation (22);
ρ, density;
τ, dimensionless independent variable, $= t$;
$^{-}$, denotes lower phase;
$*$, denotes dimensional variable.

INTRODUCTION

NUMEROUS technical problems involve the movement of a phase boundary induced by the diffusion of energy or mass. Common examples involving the conduction of heat are the solidification of castings, the thawing of permafrost, the freezing of foods, and the aerodynamic heating of missiles. Correspondingly, processes such as the dissolution of bubbles or solid particles and the tarnishing of metal surfaces involve the molecular diffusion of mass. In general, the nonlinearity associated with the moving phase boundary significantly complicates the analysis of this class of problems.

The vast majority of theoretical work in this area has been limited to the analysis of one-dimensional transport. Several comprehensive surveys of the mathematical techniques that are available for the analysis of the one-dimensional case have been published [1, 2], and a further review of this area will not be given here. The present study is concerned with the solution of moving boundary problems where two spatial coordinates are needed to characterize the location of the phase boundary.

At the present time, only a limited number of methods are available for the solution of two-dimensional moving boundary problems. In most cases, the emphasis has been placed on a general class of two-dimensional solidification or melting problems, and the following discussion is thus given in the context of this kind of system. Because of the complexity of the unsteady two-dimensional case, most of the available solution techniques are numerical rather than analytical in nature. In general, these numerical solution methods can be grouped into two basic classes: the fixed grid-moving boundary technique and methods where the latent heat of fusion is directly incorporated into the conventional unsteady-state heat-conduction equation and the problem is reduced to a single phase heat transfer analysis.

The general class of fixed grid methods is typified by the analyses of Tien and Wilkes [3], Lazaridis [4], and Springer and Olson [5]. These investigators solved two-dimensional unsteady-state heat-transfer problems by considering the movement of the phase interface through a fixed finite-difference grid. Standard finite-difference formulations could then be imposed for all nodal points sufficiently far from the interface, and special numerical schemes were devised for points in the vicinity of the moving phase boundary. Once these finite-difference approximations were formulated, conventional numerical solution methods were employed to describe the temperature field and to determine the location of the phase interface.

In the second class of solution methods, the presence of the moving phase boundary is not considered explicitly in the finite-difference solution. In the technique presented by Doherty [6], the onset of the phase change in any cell of the finite-difference grid is indicated when the temperatures at two successive time increments straddle the phase transition temperature. When the onset of melting is detected, the cell temperature is set equal to the melting temperature, and the energy input to the cell from neighboring cells is accumulated into an internal energy variable. The cell temperature is not permitted to exceed the transition point until the accumulated energy is equal to the latent heat of the material in the cell. Also in this class of methods, Hashemi and Sliepcevich [7] have presented an implicit alternating direction technique in which the heat of fusion is accounted for by an effective heat capacity which is introduced by assuming that the phase transition occurs in a finite temperature interval. Finally, Allen and Severn [8] have presented an analysis in which the heat of fusion is treated as a heat of generation term in the energy equation.

The two broad classes of solution techniques discussed above represent the most prevalent approaches for solving multidimensional unsteady-state heat-conduction problems where a phase change occurs. However, several other methods have also been proposed. Poots [9] used a weighted residual technique to consider the problem of solidification of a liquid square initially at its melting temperature. The solution was carried out by introducing assumed functional forms for the general shape of the interface and for the temperature distribution in the region. Sikarskie and Boley [10] devised a method of converting the boundary value problem with a partial differential equation into a set of integro-differential equations which could be solved by numerical or series methods. Finally, Rathjen and Jiji [11] obtained a solution to heat conduction with melting or freezing in a two-dimensional corner by treating the latent heat as a moving heat source. A nonlinear, singular integro-differential equation was derived for the interface position and an approximate solution was obtained.

Of the methods listed above, those based on standard finite-difference techniques are the most promising since they are applicable to a wide variety of diffusion-controlled moving boundary problems. As in the case of all methods of solution, the major difficulty inherent in the finite-difference solution of two-dimensional phase change problems is that associated with the time dependency and irregular nature of the surface which constitutes the phase interface. Resolution of this difficulty forms the basis for the present paper. Specifically, it is apparent that an immobilization of the phase boundary by an appropriate coordinate transformation would greatly reduce the difficulties associated with the solution of the problem.

Immobilization techniques have been developed and widely employed in the analysis of one-dimensional moving boundary problems [1]. In this paper, a coordinate transformation for the immobilization of a two-dimensional moving boundary is proposed. The basic philosophy of this approach is to simplify the numerical analysis by transforming the irregular moving boundary to a fixed boundary of simple geometry at the expense of complicating the governing partial differential equations. Since standard finite-difference techniques can readily handle complex partial differential equations but are difficult to adapt to moving and/or irregular boundaries, the transformation technique casts the problem into a form which utilizes the strength of finite-difference techniques while at the same time minimizes their shortcomings.

In order to demonstrate the application of this method, a test problem involving the solidification or melting of a two-phase finite cylinder is analyzed, and results for a range of conditions are presented. This particular example problem was chosen for several reasons:

(a) The final steady-state position of the interface for this case can be established by an analytical solution for the special condition when the thermal conductivities of the two phases are equal. Consequently, the accuracy of the developed technique can be checked at the steady-state limit.

(b) With a change in the initial conditions and other parameters, the general nature of the problem can be significantly altered so that the developed technique can be tested over a wide range of conditions.

(c) Although the example problem is sufficiently flexible to permit examination of the developed method, it is not unduly complex and the results can be simply presented.

FORMULATION OF EXAMPLE PROBLEM

The problem chosen for illustrative purposes involves two-dimensional unsteady-state heat conduction in a cylindrical container of length L and radius R. This container is filled with a material which exhibits a liquid–solid phase transition at a fusion temperature, T_f. Initially this system is at a steady-state condition with the upper surface of the cylinder maintained at the temperature T_1 and the lower surface at T_2; the outer curved surface is thermally insulated. Under these conditions, there is symmetry in the azimuthal direction and a schematic diagram of a cross section of the cylinder is presented in Fig. 1(a). If the fusion temperature, T_f, lies between T_1 and T_2, the cylinder initially contains both liquid and solid phases with a phase boundary at $z^* = H_0^*$. In the following development, lower phase quantities will be denoted by a bar ($\bar{\ }$). At time $t^* = 0$ the insulation is removed and the wall of the cylinder is exposed to a surrounding phase which is maintained at the temperature T_3. Depending on the relative value of T_3 with respect to the fusion temperature, T_f, the material inside the cylindrical container will start to solidify or melt and the phase boundary $H^*(t^*, r^*)$ will begin to move.

In the mathematical analysis of this problem the following assumptions are employed:

(a) All physical properties of both solid and liquid phases are constant.

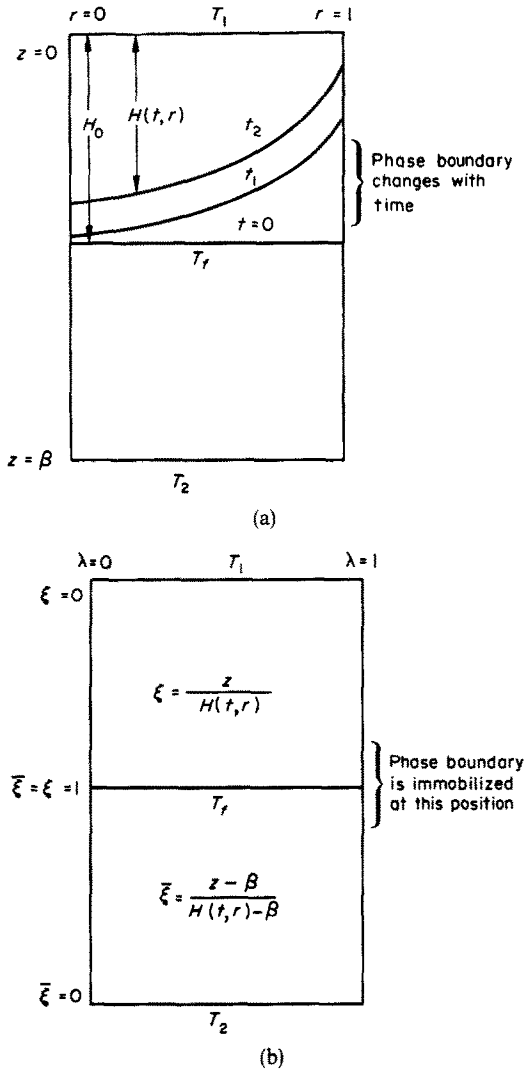

FIG. 1. Schematic diagram of example problem. (a) Conventional cylindrical coordinate system (r, z, t). (b) Immobilized coordinate systems (λ, ξ, τ) and $(\lambda, \bar{\xi}, \tau)$.

(b) The densities of solid and liquid phases are equal. Hence, it is reasonable to assume that the velocity is zero everywhere in the system.

(c) Gravitational effects can be neglected so that the system is effectively at constant pressure. Thus, the temperature is invariant on the solid–liquid interface.

(d) The container consists of very thin walls and its thermal properties can be neglected.

(e) Heat exchange between the cylinder and the surroundings can be represented by a constant heat-transfer coefficient, h.

Application of appropriate conservation laws and jump conditions in conjunction with the above assumptions yields the following dimensionless set of equations for this heat conduction-controlled moving boundary problem:

Upper phase energy equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}. \tag{1}$$

Lower phase energy equation

$$\frac{\partial \bar{T}}{\partial t} = \zeta\left(\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} + \frac{\partial^2 \bar{T}}{\partial z^2}\right). \tag{2}$$

Jump condition for energy

$$\theta\frac{\partial H}{\partial t} = \left[1 + \left(\frac{\partial H}{\partial r}\right)^2\right]\left[\left(\frac{\partial T}{\partial z}\right)_{z=H} - \kappa\left(\frac{\partial \bar{T}}{\partial z}\right)_{z=H}\right]. \tag{3}$$

Initial conditions

$$H(0, r) = H_0 = \frac{\beta\delta}{\delta + \kappa - \kappa\delta}, \quad 0 \leqslant r \leqslant 1 \tag{4}$$

$$T(0, r, z) = \delta\frac{z}{H_0}, \quad 0 \leqslant r \leqslant 1, \ 0 \leqslant z \leqslant H_0 \tag{5}$$

$$\bar{T}(0, r, z) = (\delta - 1)\left(\frac{z - \beta}{H_0 - \beta}\right) + 1, \quad 0 \leqslant r \leqslant 1, \ H_0 \leqslant z \leqslant \beta. \tag{6}$$

Upper phase boundary conditions

$$T(t, r, 0) = 0, \quad 0 \leqslant r \leqslant 1 \tag{7}$$

$$T[t, r, H(t, r)] = \delta, \quad 0 \leqslant r \leqslant 1 \tag{8}$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=1} = -\gamma[T(t, 1, z) - \varepsilon], \quad 0 < z < H(t, 1) \tag{9}$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=0} = 0, \quad 0 \leqslant z < H(t, 0). \tag{10}$$

Lower phase boundary conditions

$$\bar{T}(t, r, \beta) = 1, \quad 0 \leqslant r \leqslant 1 \tag{11}$$

$$\bar{T}[t, r, H(t, r)] = \delta, \quad 0 \leqslant r \leqslant 1 \tag{12}$$

$$\left(\frac{\partial \bar{T}}{\partial r}\right)_{r=1} = -\frac{\gamma}{\kappa}[\bar{T}(t, 1, z) - \varepsilon], \quad H(t, 1) < z < \beta \tag{13}$$

$$\left(\frac{\partial \bar{T}}{\partial r}\right)_{r=0} = 0, \quad H(t, 0) < z \leqslant \beta. \tag{14}$$

Boundary conditions for jump equation

$$\left(\frac{\partial H}{\partial r}\right)_{r=0} = 0 \tag{15}$$

$$\text{If } \delta \quad \varepsilon, \quad \frac{\partial H}{\partial r} = \frac{\gamma(\delta - \varepsilon)}{\left(\frac{\partial T}{\partial z}\right)_{z=H}}, \quad r = 1. \tag{16}$$

$$\text{If } \delta < \varepsilon, \quad \frac{\partial H}{\partial r} = \frac{\gamma(\delta - \varepsilon)}{\kappa\left(\frac{\partial \bar{T}}{\partial z}\right)_{z=H}}, \quad r = 1. \tag{17}$$

Numerical solution of this coupled set of equations is complicated by the fact that the phase boundary will move with time and the interface surface will not, in general, correspond to a coordinate surface in the conventional cylindrical coordinate system. The complexity of the numerical analysis can be greatly reduced if the problem is transformed to a coordinate system in which the phase boundary is stationary and coincides with one of the coordinate surfaces. There are numerous specific transformations which show these desired characteristics, and there are some advantages in using different transformations for the upper and lower phases. In general, the upper phase can be transformed from the (r, z, t) cylindrical coordinate system to a new system (λ, ξ, τ) where ξ is an appropriate function $\xi(r, z, t)$ and $\lambda = r$, $\tau = t$. Similarly, the lower phase can be transformed to the system $(\lambda, \bar{\xi}, \tau)$ where $\bar{\xi}$ can be a different function of r, z and t. For transformations of this general form, equations (1), (2), and (3) are converted to the following set of equations:

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial \lambda^2} + 2\frac{\partial \xi}{\partial r}\frac{\partial^2 T}{\partial \xi \partial \lambda} + \frac{\partial^2 T}{\partial \xi^2}\left[\left(\frac{\partial \xi}{\partial z}\right)^2 + \left(\frac{\partial \xi}{\partial r}\right)^2\right] + \frac{\partial T}{\partial \xi}\left[\frac{\partial^2 \xi}{\partial r^2} + \frac{\partial^2 \xi}{\partial z^2} + \frac{1}{\lambda}\frac{\partial \xi}{\partial r} - \frac{\partial \xi}{\partial t}\right] + \frac{1}{\lambda}\frac{\partial T}{\partial \lambda} \tag{18}$$

$$\frac{1}{\zeta}\frac{\partial \bar{T}}{\partial \tau} = \frac{\partial^2 \bar{T}}{\partial \lambda^2} + 2\frac{\partial \bar{\xi}}{\partial r}\frac{\partial^2 \bar{T}}{\partial \bar{\xi} \partial \lambda} + \frac{\partial^2 \bar{T}}{\partial \bar{\xi}^2}\left[\left(\frac{\partial \bar{\xi}}{\partial z}\right)^2 + \left(\frac{\partial \bar{\xi}}{\partial r}\right)^2\right] + \frac{\partial \bar{T}}{\partial \bar{\xi}}\left[\frac{\partial^2 \bar{\xi}}{\partial r^2} + \frac{\partial^2 \bar{\xi}}{\partial z^2} + \frac{1}{\lambda}\frac{\partial \bar{\xi}}{\partial r} - \frac{1}{\zeta}\frac{\partial \bar{\xi}}{\partial t}\right] + \frac{1}{\lambda}\frac{\partial \bar{T}}{\partial \lambda} \tag{19}$$

$$\theta\frac{\partial H}{\partial \tau} = \left[1 + \left(\frac{\partial H}{\partial \lambda}\right)^2\right]\left[\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial z} - \kappa\frac{\partial \bar{T}}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial z}\right]_{z=H} \tag{20}$$

It is clear that equations (18) and (19) are parabolic everywhere in their domains of definition.

The specific transformations utilized in this study are the following:

$$\xi = \frac{z}{H(t, r)} \tag{21}$$

$$\bar{\xi} = \frac{z - \beta}{H(t, r) - \beta}. \tag{22}$$

The general characteristics of these transformations are presented in Fig. 1(b). Clearly, the problem is transformed to a coordinate system where the phase boundary is defined by the surfaces $\xi = \bar{\xi} = 1$. Utilization of this coordinate transformation in equations (4)–(17) and equations (18)–(20) gives the following set of equations which completely describes the illustrative problem:

Upper phase energy equation

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial T}{\partial \lambda} - \frac{2\xi}{H}\frac{\partial H}{\partial \lambda}\frac{\partial^2 T}{\partial \xi \partial \lambda} + \frac{\partial^2 T}{\partial \xi^2}\left[\frac{1}{H^2} + \frac{\xi^2}{H^2}\left(\frac{\partial H}{\partial \lambda}\right)^2\right] + \frac{\partial T}{\partial \xi}\left[\frac{2\xi}{H^2}\left(\frac{\partial H}{\partial \lambda}\right)^2 - \frac{\xi}{H}\frac{\partial^2 H}{\partial \lambda^2} - \frac{\xi}{\lambda H}\frac{\partial H}{\partial \lambda} + \frac{\xi}{H}\frac{\partial H}{\partial \tau}\right]. \tag{23}$$

Lower phase energy equation

$$\frac{1}{\zeta}\frac{\partial \bar{T}}{\partial \tau} = \frac{\partial^2 \bar{T}}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial \bar{T}}{\partial \lambda} - \frac{2\bar{\xi}}{H - \beta}\frac{\partial H}{\partial \lambda}\frac{\partial^2 \bar{T}}{\partial \bar{\xi} \partial \lambda} + \frac{\partial^2 \bar{T}}{\partial \bar{\xi}^2}\left[\frac{1}{(H - \beta)^2} + \frac{\bar{\xi}^2}{(H - \beta)^2}\left(\frac{\partial H}{\partial \lambda}\right)^2\right] + \frac{\partial \bar{T}}{\partial \bar{\xi}}\left[\frac{2\bar{\xi}}{(H - \beta)^2}\left(\frac{\partial H}{\partial \lambda}\right)^2 - \frac{\bar{\xi}}{H - \beta}\frac{\partial^2 H}{\partial \lambda^2} - \frac{\bar{\xi}}{\lambda(H - \beta)}\frac{\partial H}{\partial \lambda} + \frac{\bar{\xi}}{\zeta(H - \beta)}\frac{\partial H}{\partial \tau}\right] \tag{24}$$

Jump condition for energy

$$\theta\frac{\partial H}{\partial\tau}=\left[1+\left(\frac{\partial H}{\partial\lambda}\right)^2\right]\times\left[\frac{1}{H}\left(\frac{\partial T}{\partial\xi}\right)_{\xi=1}-\frac{\kappa^{\cdot}}{H-\beta}\left(\frac{\partial\bar{T}}{\partial\bar{\xi}}\right)_{\bar{\xi}=1}\right]. \quad (25)$$

Initial conditions

$$H(0,\lambda)=\frac{\beta\delta}{\delta+\kappa-\kappa\delta},\quad 0\leqslant\lambda\leqslant 1 \quad (26)$$

$$T(0,\lambda,\xi)=\delta\xi,\quad 0\leqslant\lambda\leqslant 1,\ 0\leqslant\xi\leqslant 1 \quad (27)$$

$$\bar{T}(0,\lambda,\bar{\xi})=(\delta-1)\bar{\xi}+1,\quad 0\leqslant\lambda\leqslant 1,\ 0\leqslant\bar{\xi}\leqslant 1. \quad (28)$$

Upper phase boundary conditions

$$T(\tau,\lambda,0)=0,\quad 0\leqslant\lambda\leqslant 1 \quad (29)$$

$$T(\tau,\lambda,1)=\delta,\quad 0\leqslant\lambda\leqslant 1 \quad (30)$$

$$\frac{\xi}{H}\frac{\partial H}{\partial\lambda}\frac{\partial T}{\partial\xi}-\frac{\partial T}{\partial\lambda}=\gamma(T-\varepsilon),\quad \lambda=1,\ 0<\xi<1 \quad (31)$$

$$\left(\frac{\partial T}{\partial\lambda}\right)_{\lambda=0}=0,\quad 0\leqslant\xi<1. \quad (32)$$

Lower phase boundary conditions

$$\bar{T}(\tau,\lambda,0)=1,\quad 0\leqslant\lambda\leqslant 1 \quad (33)$$

$$\bar{T}(\tau,\lambda,1)=\delta,\quad 0\leqslant\lambda\leqslant 1 \quad (34)$$

$$\frac{\bar{\xi}}{H-\beta}\frac{\partial\bar{T}}{\partial\bar{\xi}}\frac{\partial H}{\partial\lambda}-\frac{\partial\bar{T}}{\partial\lambda}=\frac{\gamma}{\kappa}(\bar{T}-\varepsilon),\quad \lambda=1,\ 0<\bar{\xi}<1 \quad (35)$$

$$\left(\frac{\partial\bar{T}}{\partial\lambda}\right)_{\lambda=0}=0,\quad 0\leqslant\bar{\xi}<1. \quad (36)$$

Boundary conditions for jump equation

$$\left(\frac{\partial H}{\partial\lambda}\right)_{\lambda=0}=0. \quad (37)$$

If $\delta>\varepsilon$,

$$\frac{\partial H}{\partial\lambda}=\frac{H\gamma(\delta-\varepsilon)}{\left(\frac{\partial T}{\partial\xi}\right)_{\xi=1}},\quad \lambda=1 \quad (38)$$

If $\delta<\varepsilon$,

$$\frac{\partial H}{\partial\lambda}=\frac{\gamma(H-\beta)(\delta-\varepsilon)}{\kappa\left(\frac{\partial\bar{T}}{\partial\bar{\xi}}\right)_{\bar{\xi}=1}},\quad \lambda=1. \quad (39)$$

FINITE-DIFFERENCE SOLUTION OF EXAMPLE PROBLEM

The finite-difference analysis of the example problem involves consideration of two parabolic, second-order equations, equations (23) and (24), and a first-order differential equation, equation (25). These equations are more complex than the equations of the original problem formulation, but they can be used with regular, fixed finite-difference grids in both upper and lower phases. Central differences were used for all spatial derivatives and backward differences for all time derivatives, so that implicit forms of all of the difference equations were obtained. Since equations (23) and (24) explicitly include the boundary position, it is not convenient to utilize an implicit alternating direction method to solve for T and $\bar{T}$. Therefore, it was necessary to employ iterative methods in the solutions of the finite-difference forms of equations (23) and (24) in addition to the iterative scheme used to solve the difference equation derived from equation (25).

Once the location of the phase boundary and the temperature arrays were set according to the initial conditions, equations (26)–(28), the sequence of computations repeated over successive increments of time (τ) was as follows:

(a) Solve for a new temperature distribution in the upper phase consistent with the current boundary position using the implicit difference form of equation (23). New values of temperature are calculated along each line in the radial (λ) direction by utilization of a Gaussian elimination method to solve a tridiagonal matrix system.

(b) Repeat step (a) until the temperature array in the upper phase has converged.

(c) Solve for a new temperature distribution in the lower phase consistent with the current boundary position using the method detailed in step (a) and an implicit difference form of equation (24).

(d) Repeat step (c) until the temperature array in the lower phase has converged.

(e) Solve for a new boundary position consistent with the current temperature arrays using the implicit difference form of equation (25). This step involves an iterative scheme since equation (25) is nonlinear in H.

(f) Repeat steps (a)–(e) until all arrays have converged for the new value of τ. Each time step involves three iterative schemes imbedded inside an overall iteration loop.

A variable time step was used in the solution so that smaller time steps could be utilized at early times when the temperature gradients are large. For a typical run, thirty spatial increments were used in the radial direction and fifteen in each phase in the axial direction. The smallest time step used was $2{\cdot}5\times10^{-5}$. Mesh sizes for the radial, axial, and time variables were varied in the usual manner in order to establish the convergence of the finite-difference solution to the solution of the differential equations.

No stability criterion was established for the implicit difference equations, but no difficulties were encountered for all time steps utilized in this study.

ANALYTICAL STEADY-STATE SOLUTION

When the thermal conductivities of the solid and liquid phases are equal ($\kappa=1$), the steady-state limit of the transient problem reduces to the analysis of steady conduction in a homogeneous cylinder. For this special case, the steady temperature field is described by the following set of equations:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=0 \quad (40)$$

$$T(r,0)=0,\quad 0\leqslant r\leqslant 1 \quad (41)$$

$$T(r, \beta) = 1, \quad 0 \leqslant r \leqslant 1 \tag{42}$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=1} = -\gamma[T(1, z)-\varepsilon], \quad 0 < z < \beta \tag{43}$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=0} = 0, \quad 0 \leqslant z \leqslant \beta. \tag{44}$$

The solution to these equations is simply

$$T(r, z) = \frac{z}{\beta} + \sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi r}{\beta}\right) \sin\frac{n\pi z}{\beta} \tag{45}$$

where

$$A_n = \frac{\dfrac{2\gamma}{n\pi}[\varepsilon+(1-\varepsilon)(-1)^n]}{\dfrac{n\pi}{\beta} I_1\left(\dfrac{n\pi}{\beta}\right)+\gamma I_0\left(\dfrac{n\pi}{\beta}\right)}. \tag{46}$$

Equation (45) can be used to locate the steady-state position of the phase interface by determining the locus of points where the temperature is equal to the dimensionless fusion temperature δ.

RESULTS AND DISCUSSION

Several variations of the example problem were solved in order to demonstrate the applicability of the immobilization transformation and the associated finite-difference method developed in this paper. The parameters in the set of equations describing heat transfer in the cylinder are β, γ, δ, ε, ζ, θ, and κ. In all solutions that were obtained, β, δ, and ε were held constant, and thus a cylinder of fixed geometry and fixed fusion and ambient temperatures was considered. Consequently, the effects of the thermal properties of the phases (ζ and κ), of the Nusselt number (γ), and of the ratio of latent to sensible heat (θ) on the movement of the phase boundary were investigated.

Results for five cases are presented in this paper, and the values of the parameters used in these cases are listed in Table 1. These results describe either melting or solidification in the cylinder, depending on the relative values of T_1 and T_2. For $T_2 > T_1$, the lower phase is liquid and, since $T_3 > T_f$, melting of the upper phase takes place. For $T_1 > T_2$, the lower phase is solid and, since $T_3 < T_f$, freezing of the upper phase occurs.

One test of the accuracy of the solutions obtained by the present method is a comparison of the steady-state boundary position calculated from a finite-difference solution with that calculated from the analytical solution (equation (45)) for the case $\kappa = 1$. This comparison is given in Table 2 for case A. The difference between the analytical and finite-difference solutions is approximately 0·5 per cent at all radial positions except $r = 1$ where the discrepancy is about 1 per cent. This excellent agreement provides one indication of the type of accuracy that might be expected from the proposed method.

Table 2. Comparison of steady-state finite-difference solution for case A with analytical solution

r	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
$H(\infty, r)$, Analytical solution	0·4687	0·4679	0·4654	0·4611	0·4546	0·4451	0·4317	0·4125	0·3845	0·3419	0·2737
$H(\infty, r)$, Finite-difference solution	0·4710	0·4701	0·4676	0·4631	0·4564	0·4466	0·4328	0·4131	0·3845	0·3412	0·2711

Table 1. Characteristics of cases investigated

Case*	κ	ζ	γ	θ
A	1·0	1·0	3·0	1·5
B	1·0	1·0	3·0	0·5
C	2·0	2·0	1·5	1·5
D	2·0	2·0	3·0	1·5
E	2·0	2·0	4·5	1·5

*For cases A–E the following parameters were used: $M = 30$, $N = 15$, $\delta = 0·5$, $\varepsilon = 0·75$, $\beta = 1·0$.

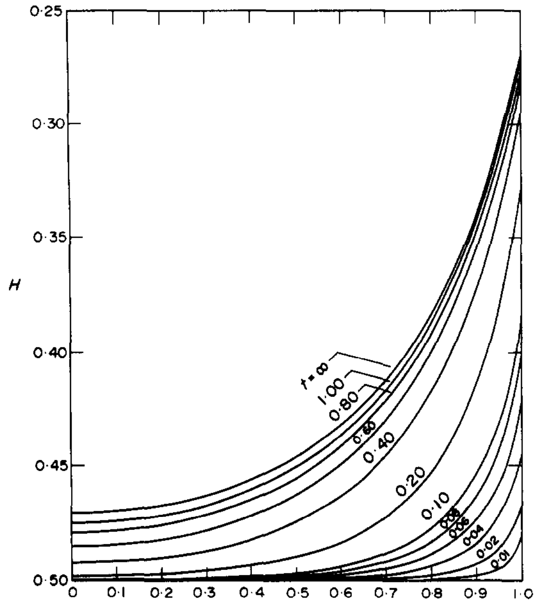

FIG. 2. Boundary position for case A.

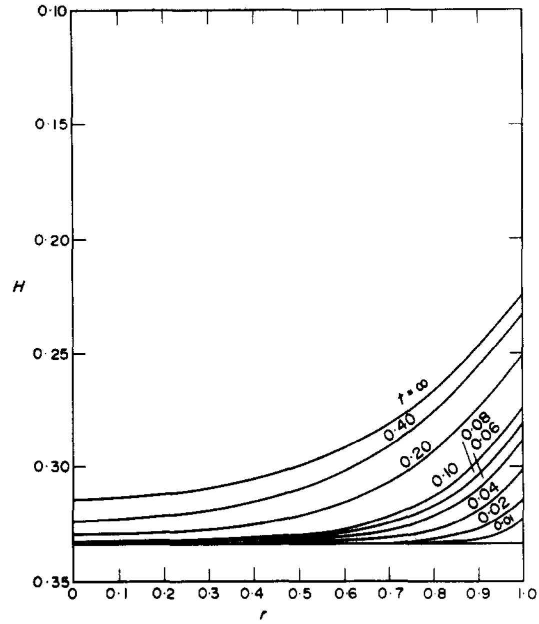

FIG. 4. Boundary position for case C.

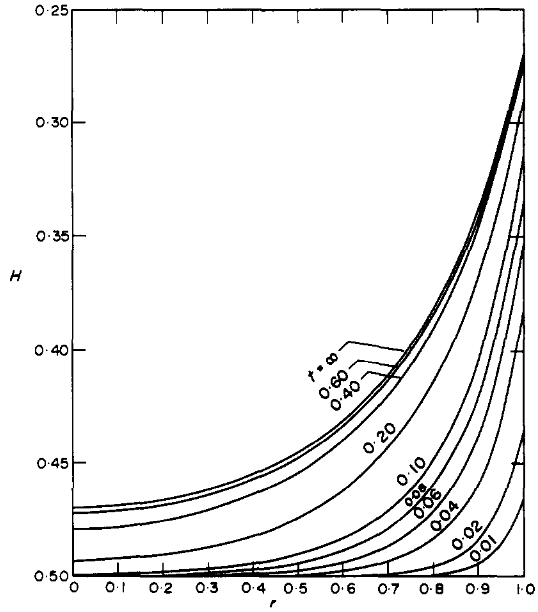

FIG. 3. Boundary position for case B.

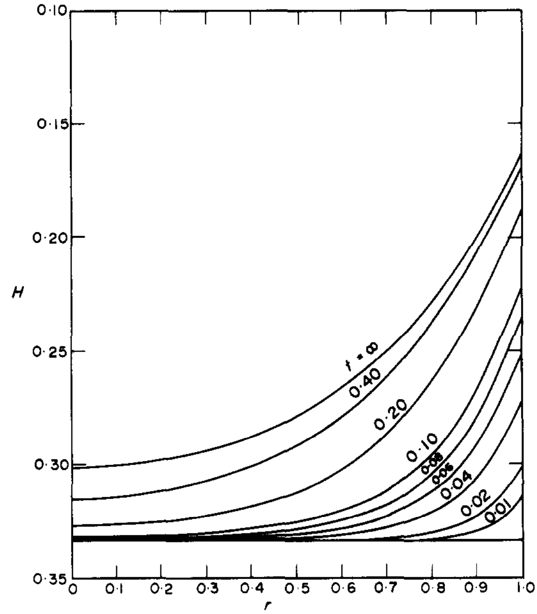

FIG. 5. Boundary position for case D.

Finite-difference results for cases A–E are presented in Figs. 2–6 in terms of the position of the phase boundary, $H(t, r)$. All five cases considered show the same qualitative trends. The phase boundary movement is of course appreciable initially only near $r = 1$, but significant boundary displacement is eventually initiated throughout the cylindrical region. The velocity of the phase interface decreases monotonously with time as the steady-state position is approached asymptotically.

It is of some interest to examine how θ, ζ and κ, the thermal property ratios, and γ, the Nusselt number, affect the velocity and steady-state position of the phase interface. In Fig. 7, the position of the phase boundary at the outer cylindrical surface, $H(r = 1)$, is plotted as a function of dimensionless time for cases A and B for which all parameters except θ are identical. The steady-state boundary position is approached significantly faster for case B for which θ has one-third of the value used in case A, but the limiting boundary surface is the same for the two cases. The above behavior is what would be expected from consideration of equation (3).

In Fig. 8, the time variations of $H(r = 1)$ for cases A and D are compared. These cases are identical except for the thermal property parameters, ζ and κ. Since the

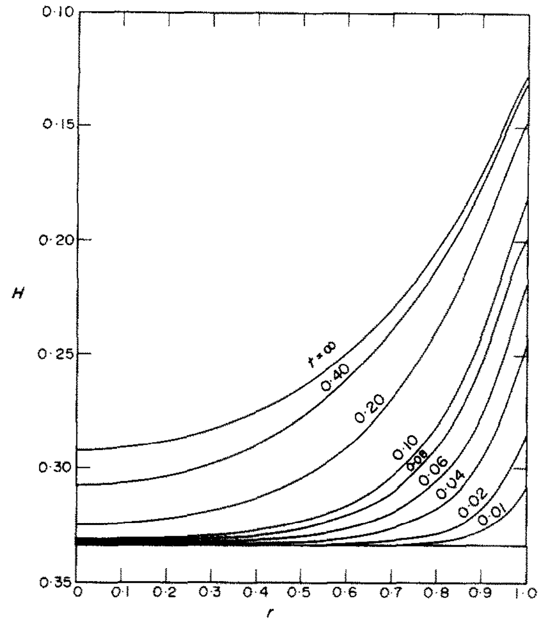

FIG. 6. Boundary position for case E.

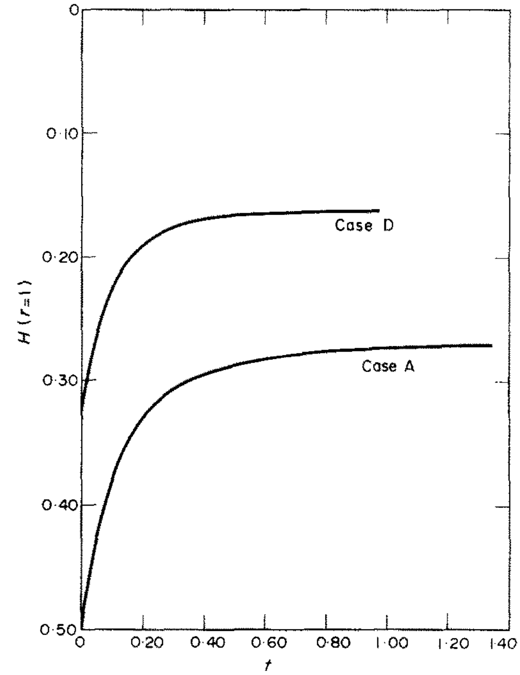

FIG. 8. Time variation of boundary position at $r = 1$ for cases A and D.

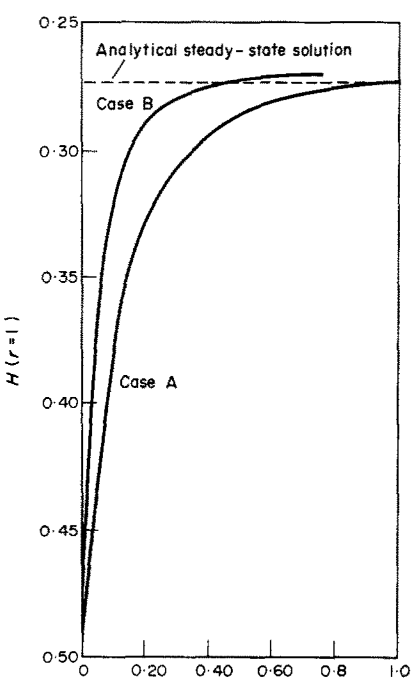

FIG. 7. Time variation of boundary position at $r = 1$ for cases A and B.

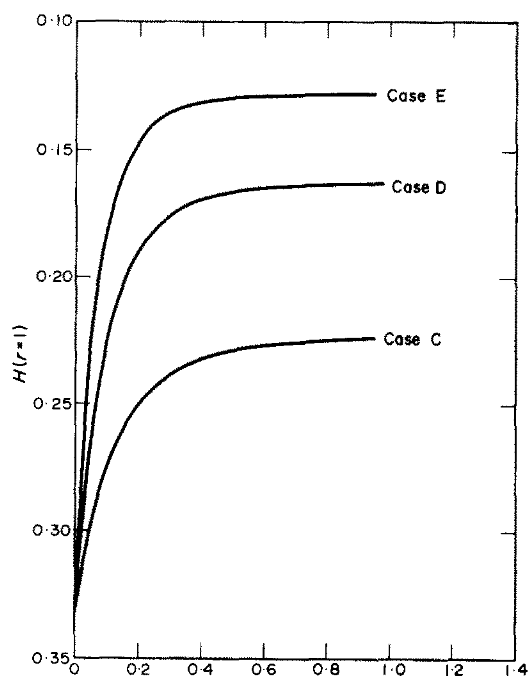

FIG. 9. Time variation of boundary position at $r = 1$ for cases C, D, and E.

densities of the phases have been assumed equal, the thermal properties of the phases must be identical for case A, whereas the heat capacities, but not the thermal conductivities, are the same for case D. Since the thermal conductivity of the lower phase for case D is twice that for case A, there should be generally higher velocities and greater steady-state displacement of the phase boundary for case D because of the increased heat transfer. The quantitative nature of the enhanced heat transfer is shown in Fig. 8, where it can be seen that the final position of $H(r = 1)$ for case D is closer to the top of the cylinder than the steady-state position for case A.

Cases C, D, and E show the effect of increasing the Nusselt number, γ, on the movement of the phase boundary. Again, as can be seen from Fig. 9, the increased heat transfer between the cylinder and the surroundings gives greater ultimate displacement of the

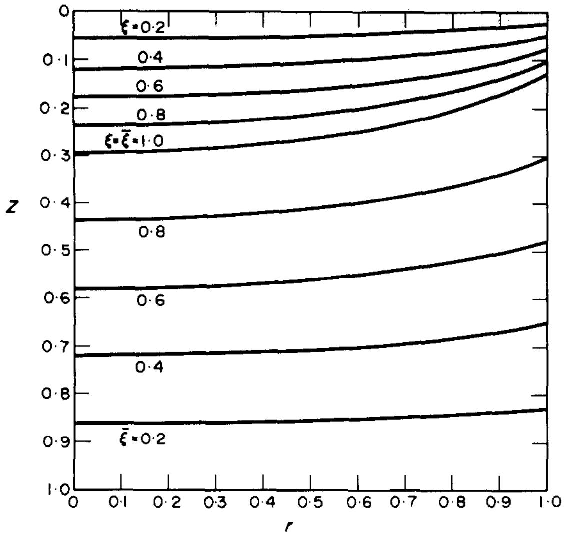

FIG. 10. λ coordinate lines at steady state for case E.

solid–liquid interface and a faster approach to the steady-state limit.

Finally, we consider a graphical comparison of the transformed coordinate systems, (λ, ξ) and $(\lambda, \bar{\xi})$, with the conventional cylindrical coordinate system (r, z). The surfaces denoting angular position are of course the same in the two coordinate systems. Such a comparison is given in Fig. 10 for case E at the steady-state limit. In this figure, the λ coordinate lines, the intersections of the ξ or $\bar{\xi}$ surfaces with any plane representing constant angular position, are shown imposed on a cylindrical coordinate system for various values of ξ and $\bar{\xi}$. Since the surfaces ξ = constant and $\bar{\xi}$ = constant are not parallel to the planes z = constant, the λ coordinate lines show significant curvature, in contrast to the r coordinate lines which are parallel rays passing through the z axis. The curvature of these coordinate lines is of course directly dependent on the curvature of the phase interface, the coordinate line for the surfaces $\xi = \bar{\xi} = 1$.

The boundary immobilization technique proposed in this paper has been illustrated by examining an unsteady heat conduction problem, but it is reasonable to expect that the principles can be applied to a wide variety of systems of different geometries and characteristics. For example, the analysis of moving boundary problems involving the equations of motion coupled with heat or mass transfer, such as in the study of forced or natural convection phenomena, could presumably be simplified by this technique. Indeed, mapping of time-dependent and/or irregular boundaries into fixed and/or regular surfaces should usually result in significant simplifications in the numerical solution of partial differential equations since boundary conditions are expressed in the most convenient form for numerical analysis. Finally, it should be noted that the immobilization transformation could be modified to permit flexibility in the distribution of grid points for the most accurate representation of the differential equations. Such transformations have been employed in conjunction with one-dimensional immobilization transformations [12], and they prove to be very useful in cases where large gradients exist in certain regions of the domain of interest.

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ETUDE ANALYTIQUE DES PROBLEMES BIDIMENSIONNELS DE DIFFUSION CONTROLEE AVEC FRONTIERES MOBILES

Résumé—L'article présente une méthode pour l'étude analytique de problèmes bidimensionnels instationnaires de transfert de chaleur ou de masse par diffusion, caractérisés par des frontières irrégulières en mouvement. La technique utilisée comprend une transformation qui permet d'immobiliser les frontières et un schéma numérique pour la résolution des équations transformées. Dans le cas présent, la technique d'immobilisation consiste à transformer les équations fondamentales aux dérivées partielles par un changement de coordonnées dans lesquelles les frontières de phase correspondent à des surfaces coordonnées fixées. Un exemple de problème est analysé qui fait intervenir la solidification ou la fusion d'un cylindre de longueur finie, les résultats sont présentés pour divers conditions.

ANALYTISCHE UNTERSUCHUNG ZWEIDIMENSIONALER DIFFUSIONSVORGÄNGE MIT VERÄNDERLICHEN GRENZEN

Zusammenfassung—Der Aufsatz beschreibt die analytische Behandlung des instationären, zweidimensionalen Wärme- und Stofftransports bei veränderlichen Grenzen. Das Verfahren umfaßt Transformationen und ein Schema zur numerischen Lösung der transformierten Gleichungen.

Spezielle der Übergang in den Zustand der Unbeweglichkeit besteht aus der Transformation der maßgeblichen partiellen Differentialgleichung auf ein Koordinatensystem, in dem die Phasengrenzen festen Oberflächen entsprechen. Als Beispiel werden der Erstarrungs- oder Schmelzvorgang eines Zylinders endlicher Länge analysiert und Ergebnisse für unterschiedliche Bedingungen mitgeteilt.

АНАЛИЗ ДВУМЕРНЫХ ЗАДАЧ ДИФФУЗИИ С ДВИЖУЩИМИСЯ ГРАНИЦАМИ

Аннотация — В статье представлена методика анализа нестационарных двумерных задач диффузионного тепло- или массопереноса с движущимися нерегулярными границами. Методика включает в себя преобразование иммобилизации и численную схему для решения преобразованных уравнений. В частности, иммобилизация состоит из преобразования основных дифференциальных уравнений в частных производных в координатную систему, где границы фаз соответствуют фиксированным координатным поверхностям. В качестве примера анализируется задача затвердевания или плавления цилиндра конечной длины, представлены результаты для ряда условий.